On how a joint interaction of two innocent partners (smooth advection & linear damping) produces a strong intermittency

M. Chertkov
Physics Department, Princeton University, Princeton, NJ 08544
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Forced advection of passive scalar by a smooth d-dimensional incompressible velocity in the presence of a linear damping is studied. Acting separately advection & dumping do not lead to an essential intermittency of the steady scalar statistics, while being mixed together produce a very strong non-Gaussianity in the convective range: q-th (positive) moment of the absolute value of scalar difference, $\langle|\theta(t;\mathbf{r})-\theta(t;0)|^q\rangle$ is proportional to r^{ξ_q}, $\xi_q=\sqrt{d^2/4+\alpha dq/[(d-1)D]}-d/2$, where α/D measures the rate of the damping in the units of the stretching rate. Probability density function (PDF) of the scalar difference is also found.

Advection of passive scalar $\theta(t;\mathbf{r})$ by incompressible velocity field is a classical problem in turbulence theory. The problem attracts a lot of recent attention for remarkable combination of a rich physics and nontrivial rigorous results derived. R.H. Kraichnan pioneered the rigorous study of the problem inventing the temporal short-correlated but spatially non-smooth model of velocity for which the simultaneous pair correlation function of the scalar was found [1]. However, the question of possible anomalous behavior of higher order ($n>1$) structure functions $S_{2n}(r)=\left\langle(\theta(t;r)-\theta(t;0))^{2n}\right\rangle\sim r^{\xi_{2n}}$ was posed only 25 years later [2]. Next, the anomalous scaling $\Delta_{2n}\equiv n\xi_2-\xi_{2n}$, describing the law of the algebraic growth with L/r (where L is the scale of the scalar pumping) of the dimensionless ratio $S_{2n}(r)/\left[S_2(r)\right]^n$, was shown to exist generically [3–5]. The anomalous exponent was calculated perturbatively in expansions about three non-anomalous ($\Delta_{2n}=0$) limits, of large space dimensionality d [3,6], of extremely non-smooth [4,7] and almost smooth [5] velocities respectively. A strong anomalous scaling (saturation of ξ_{2n} to a constant) was found for the Kraichnan model at the largest n by a steepest descent formalism [8]. Although the restricted asymptotic information about anomalous exponent in the model is available a future possibility to establish a whole rigorous dependence of ξ_{2n} on n, d and degree of velocity non-smoothness seems very unlikely (in a sense, the recent Lagrangian numerics [9] on the ζ_4 as a function of ζ_2 at $d=3$ compensates the lack of rigorous information).

In the present letter I discuss yet another passive scalar model with nontrivial anomalous behavior, $\zeta_{2n}<n\xi_2$, which is possible to resolve explicitly for all the values of the governing parameters. The model describes generalization of the smooth (Batchelor) limit of the Kraichnan model on the case of a linear damping of the scalar. The pure Batchelor model (no damping), studied detaily in [10–17], shows non-anomalous (logarithmic and almost Gaussian for the Gaussian form of the pumping) behavior. The limit of a huge linear damping (neglect advection) is also non-anomalous. However, as it will be shown below (see (6,7)), the anomalous scaling does exist generically, and appears to be a nontrivial function of n, d and a parameter standing for the damping-to-convection ratio. The model describes forced advection of a scalar pollutant in the viscous-convective range (the viscous-to-diffusivity ratio is supposed to be large) absorbed instantly and homogeneously, for example via a chemical reaction with other species present in the flow. Therefore, all the predictions of the letter may be checked experimentally as well as numerically.

Consider advection of passive scalar $\theta(t;\mathbf{r})$ by a smooth incompressible velocity field, $\mathbf{u}(t;\mathbf{r})=\hat{\sigma}(t)\mathbf{r}$ ($\hat{\sigma}(t)$ is $d\times d$ traceless matrix) in a presence of a linear damping and diffusion,

$$\partial_t\theta+\sigma^{\mu\nu}(t)r^{\mu}\nabla_r^{\nu}\theta=\kappa\triangle_r\theta-\alpha\theta+\phi. \tag{1}$$

The scalar is forced by random field $\phi(t;\mathbf{r})$, which for a sake of simplicity is considered to be Gaussian thus fixed completely by $\langle\phi(t_1;\mathbf{r}_1)\phi(t_2;\mathbf{r}_2)\rangle=\chi(|\mathbf{r}_1-\mathbf{r}_2|)\delta(t_1-t_2)$, where the function $\chi(r)$ decays fast enough if r exceeds the integral scale L. Although we are sure that many principal results of the paper can be generalized for the case of a finite temporal correlations of the strain matrix $\hat{\sigma}$, only the short correlated one allowing a simple derivation is considered here. The Gaussian statistics of $\hat{\sigma}$ is fixed by the pair correlation function $\left\langle\sigma^{\eta\mu}(t)\sigma^{\beta\nu}(t')\right\rangle$ equal to

$$D\left[(d+1)\delta^{\mu\nu}\delta^{\eta\beta}-\delta^{\mu\eta}\delta^{\nu\beta}-\delta^{\mu\beta}\delta^{\nu\eta}\right]\delta(t-t'). \tag{2}$$

We start studying the pair correlation function of the scalar field: $F(r_{12})=\langle\theta(t;\mathbf{r}_1)\theta(t;\mathbf{r}_2)\rangle$. Averaging two replicas of (1) multiplied by the respective scalar counterparts one gets

$$\left[-r^{1-d}\partial_r r^d\left(D(d-1)r+\frac{2\kappa}{r}\right)\partial_r+2\alpha\right]F(r)=\chi(r). \tag{3}$$

Consider the case of a step-like pumping function, when $\chi(r) = P = const$ at $r < L$, and zero otherwise. Introduce a forced solution of this equation, $F_f(r) = P\vartheta(L-r)/[2\alpha]$. Two zero modes of the operator from the lhs of (3) should be added to $F_f(r)$ to respect continuity of $F(r)$ and its derivative at $r = L$. Zero mode added at the upper ($r > L$) interval should vanish at $r \to L$, while its counterpart added at $r < L$ should be finite at the origin ($r \to 0$). If the dissipative scale, $r_d = \sqrt{\kappa/\max\{D,\alpha\}}$ is small enough, one gets

$$F(r) = \frac{P}{2\alpha}\begin{cases} 1 - \frac{\xi_-}{\xi_- - \xi_+}\left(\frac{r}{L}\right)^{\xi_+}, & r_d \ll r < L, \\ \frac{\xi_+}{\xi_+ - \xi_-}\left(\frac{r}{L}\right)^{\xi_-}, & r > L \gg r_d; \end{cases} \tag{4}$$

where $\xi_\pm \equiv \pm\sqrt{d^2/4 + 2\alpha d/\left[(d-1)D\right]} - d/2$. Finite diffusivity generalization of (4) (of its zero-mode part) can be presented in terms of the hyper-geometric function. Dominant contribution into $S_2(r)$ at $r \ll L$ stems from a zero mode of the operator on the rhs of (3) scaling as r^{ξ_+}.

Come to the study of the scalar difference stationary PDF, $\mathcal{P} \equiv \langle\delta\left(x - \delta\theta_r\right)\rangle$. Generally, at zero diffusion ($\kappa \to 0$) the stationary limit is perfectly achieved via the direct balance between the pumping ϕ and the α-damping. Therefore, there is no dissipative anomaly in the case and we may simply write the Fokker-Planck equation

$$D(d-1)r^{1-d}\partial_r r^{d+1}\partial_r\mathcal{P} + \alpha\partial_x\left(x\mathcal{P}\right) = 0. \tag{5}$$

(5) is valid at $x < \theta_L$, where θ_L is the amplitude of the scalar field at the integral scale, which is estimated by $P/\max\{\alpha, D\}$. Even without calculation of the PDF itself we may simply get the anomalous exponents for the structure functions of all the orders, by means of integration of (5) against the respective moments of x. The only thing left is to pick up a vanishing at $r \to 0$ solution of the linear ordinary homogeneous differential equation (zero mode of the eddy-diffusivity operator). It gives for the even moments (odd moments are constrained to be zero due to pumping isotropy and Gaussianity)

$$\frac{S_{2n}(r)}{\theta_L^{2n}} \sim \left(\frac{r}{L}\right)^{\xi_{2n}}, \qquad \xi_{2n} = \sqrt{\frac{d^2}{4} + \frac{2\alpha dn}{(d-1)D}} - \frac{d}{2}, \tag{6}$$

where the dependence on L and θ_L is restored just by dimensionality. (6) is compatible with (4) and it holds for any d , n, and α. Notice that at the fixed value of the damping-to-advection ratio α/D and d going to infinity ξ_{2n} goes to zero.

Although the calculation of anomalous exponents was our main goal the PDF itself may be also simply extracted out of (5) if additionally the form of the PDF at the integral scale (which in its own term relates to concrete form of the pumping) is fixed. For example, the Gaussian PDF at the integral scale corresponds to

$$\begin{aligned} \mathcal{P}\left(x;r\right) &= \frac{\ln\left[L/r\right]}{\pi x\sqrt{a}}\left(\frac{L}{r}\right)^{d/2} Q\left(\frac{x}{\theta_L};\ln\left[L/r\right]\right), \\ Q\left(y;z\right) &\equiv \int\limits_0^{\ln[1/y]} \exp\left[-\frac{z^2}{4at} - \frac{d^2a}{4}t - y^2\exp\left[2t\right]\right]\frac{dt}{t^{3/2}}, \end{aligned} \tag{7}$$

where $a \equiv D(d-1)/\alpha$. At $L \gg r$ the PDF shows a change in behavior about $x_c \equiv \theta_L\left[r/L\right]^{1/(da)}$. $Q\left(y;z\right)$ is finite at the origin, $Q\left(0;z\right) \sim \left(r/L\right)^{d/2}/\ln\left[L/r\right]$, and further the alrgebraic in x decay at $x \ll x_c$, $Q\left(y;z\right) - Q\left(0;z\right) \sim y^{d^2a/4}$, turns into $Q\left(y;z\right) \sim y^{d^2a/4}\exp\left[-z^2/\left(a\ln\left[1/y\right]\right)\right]$ at $x \gg x_c$. Therefore, (6) is applicable for all the positive (not necessarily integer) moments of $|\delta\theta_r|$. All the negative moments are divergent.

The possibility of two-point consideration explained above is based on the absence of the dissipative anomaly. To prove this and also to shed some light on the dynamical origin of anomalous behavior we consider Lagrangian multi-point representation of the problem and show how does it lead to the same answer (6) for the asymptotic behavior of the structure functions. (1) is equivalent to

$$\begin{cases} \theta(t;\mathbf{r}) = \int\limits_0^\infty dt'\exp\left[-\alpha t'\right]\phi\left(t';\rho(t-t')\right) \\ \frac{d}{dt}\rho(t) = \hat{\sigma}(t)\rho(t) + \eta(t), \qquad \rho(0) = \mathbf{r}, \end{cases} \tag{8}$$

where $\eta(t)$ is the Langevin noise fixed by, $\left\langle\eta^\alpha(t)\eta^\beta(t')\right\rangle = 2\kappa\delta(t-t')$. Averaging the simultaneous product of $2n$ different replicas of (8) one gets $F_{1\cdots 2n} \equiv \langle\theta_1\cdots\theta_{2n}\rangle$. Once the average of the multi point product is known it is easy

to construct the desirable structure function $S_{2n}(r)$ fusing the $2n$ points. Moreover, one can get S_{2n} from the general object with all the points $\mathbf{r}_i$ being placed along a straight line. Also we expect, and it will be confirmed below, that at $\alpha > 0$ all the fused averages are finite also in the limit of zero diffusivity. The last observation allows to consider the infinite Peclet number $Pe \equiv L/r_d$ limit just replacing κ by zero. Therefore, for the collinear geometry $\mathbf{r}_i = \mathbf{n}\, r_i$ and at $Pe \to \infty$ direct averaging of (8) with respect to statistics of ϕ and $\hat{\sigma}$ gives

$$F_{1\cdots 2n} = \sum_{\{i_1,\cdots,i_{2n}\}}^{\{1,\cdots,2n\}} \left\langle \prod_{k=1}^{n} \int_0^\infty dt_k e^{-\alpha t_k} \chi\left[e^{\eta(t_k)} r_{i_k;i_{k+1}}\right] \right\rangle_\eta , \tag{9}$$

where $\eta(t) \equiv |\hat{W}(t)\mathbf{n}|$ and $\hat{W}(t)$ satisfies $d\hat{W}(t)/dt = \hat{\sigma}(t)\hat{W}(t)$. The $\alpha = 0$ version of (9) was calculated in [14] for the $d = 2$ case and generalized for any $d \geq 2$ in [15] via a change of variables and further straightforward transformation of the path integral standing for the average over $\hat{\sigma}(t)$. Average with respect to all the "angular" degree of freedoms gives the following effective measure of averaging with respect to the rate of stretching along the $\mathbf{n}$ direction [14,15]

$$\mathcal{D}\eta(t) \exp\left[-d \int_0^\infty dt \frac{\dot{\eta}^2 + D^2(d-1)^2 - 2(d-1)\dot{\eta}D}{4(d-1)D}\right].$$

Therefore, averaging with respect to the fluctuations of the η field produces

$$\begin{aligned} F_{1\cdots 2n} &= n! \int_0^\infty dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \int_{-\infty}^{+\infty} d\eta_1 \cdots d\eta_n \exp\left[\frac{d}{2}\eta_1 - \frac{D(d-1)d}{4} t_1\right] \\ &\times \sum_{\{k_i,\cdots,k_{2n}\}}^{\{1,\cdots,2n\}} \prod_{i=1}^{n} \left[e^{2\alpha t_i} \chi\left(e^{\eta_i} r_{k_{2i},k_{2i+1}}\right) G\left(t_{i-1} - t_i; \eta_{i-1} - \eta_i\right)\right] , \end{aligned} \tag{10}$$

where $t_{i+1} = \eta_{i+1} = 0$ and $G(t;\eta) \equiv \sqrt{\frac{\pi d}{4(d-1)tD}} \exp\left[-\frac{d\eta^2}{4(d-1)tD}\right]$ is the Green function of the diffusive kernel. The integrand of (10) decays exponentially in time, so the major contribution into the object forms at $t_i \sim 1/\alpha$, it does not depend on any r_{ij}. The first r-dependent contribution stems from $n-1$ temporal integrals formed at $\tau \sim 1/\alpha$, and one at $t_i \sim \tau_r \sim \ln[L/r]/\max\{\alpha, D\}$ (this special integration brings a spatial dependence into the object, therefore on a single distance). Generally, there exists a variety of terms with all the possible combinations, like a term with k integration formed at τ, while $n-k$ ones at τ_r, and therefore dependent explicitly on $2(n-k)$ points. However, we are looking exclusively for a term dependent on all the $2n$ points since only such a term of (10) contributes $S_{2n}(r)$. And it is really simple to calculate the scaling of the term making use of the temporal scale separation, $\tau_r \gg \tau$. Indeed, the large time contribution may be extracted out of (10) in a saddle-point calculation. Variation of all the exponential terms in (10) with respect to t_i gives a chain of saddle equations. The χ functions in the integrand of (10) limits the η integrations from above by $\ln[L/r]$. Therefore, the desirable $2n$-points contribution forms at $t_i = \sqrt{d/\left[4(d-1)D\left(2\alpha n + D(d-1)d/4\right)\right]} \ln[L/r]$, and $\eta_i = \ln[L/r]$, where in the leading logarithmic order one should not distinguish between contributions of different separations r_{ij}. Finally, substituting the saddle-point values of t_i and η_i into (10) we arrive at (6).

The basic physics of nonzero ξ_{2n} (means deviating from the naive balance of pumping and advection) and generally anomalous ($\Delta_{2n} \neq 0$) scaling at $\alpha > 0$ can be stated quite directly. According to (9) the advection changes scales but not amplitude, while the amplitude of injected scalar field decays exponentially from the time of injection at the constant rate α. The temporal integrals in (10) forms at the mean time to reach a scale which is proportional to the negative log of the scale. However, the effective spread in the factor by which amplitude has decayed, upon reaching a given scale, increases as scale decreases. It is why $\xi_{2n} > 0$. Also there is more room for fluctuations about the mean time due to the interference between the exponential decay of the scalar amplitude and fluctuations of the stretching rate η. Thus intermittency increases with decrease of scale size.

I conclude by couple of general remarks. First of all, the model gives an example of the situation when the dissipative anomaly is absent, while the intermittency (anomalous scaling, $\Delta_{2n} \neq 0$) takes place. Second, continuous dependance of the exponents on the damping rate originates from coincidence of the scaling dimensions (zero in the Batchelor case) of the bare eddy diffusivity operator and the damping-dependent correction to it.

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